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Global Exponential Stability of Delayed Fuzzy Cohen – Grossberg Neural Networks with Impulse Control

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Abstract: We investigate a class of fuzzy Cohen – Grossberg neural networks with finite or infinite distributed delays and impulse control. By employing an inequality technique, we conclude that there exists a globally exponentially stable equilibrium and the uniqueness of the equilibrium is also obtained. Our impulsive controller is more general and applicable.

Key words: fuzzy Cohen – Grossberg neural networks(FCGNNs) ; global exponential stability; distributed delay; impulse control

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一类脉冲控制的时滞模糊 Cohen – Grossberg 神经网络系统的全局指数稳定性分析

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摘要:对一类具有分布时滞和脉冲控制的模糊 Cohen – Grossberg 神经网络系统进行了研究. 运用不等式技巧讨论了该系统平衡点的存在性、唯一性和全局指数稳定性并得到了一些充分条件. 本模型的脉冲控制条件更具一般性和适用性.

关键词:模糊 Cohen – Grossberg 神经网络系统; 全局指数稳定性; 分布时滞; 脉冲控制

0 Introduction

In reality, as a cause of instability and bad performance, time delays may lead to some complex dynamic behaviors in many applications [1]. Moreover, since T. Yang and L. Yang [2] first introduced the fuzzy logic to consider the uncertainty or vagueness in image processing and pattern recognition, the stability of the fuzzy Cohen

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－Grossberg neural networks with various kinds of delays has been extensively studied [3-6].

In [7], the author considered a class of fuzzy cellular neural networks with distributed delays of the following form

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}\mu_j + I_i + \bigwedge_{j=1}^{n}\alpha_{ij}\int_{t-\tau}^{t} k_j(t-s) f_j(x_j(s))\,ds$$
$$+ \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n}\beta_{ij}\int_{t-\tau}^{t} k_j(t-s) f_j(x_j(s))\,ds + \bigvee_{j=1}^{n} H_{ij}\mu_j .$$

Exponential stability was obtained with finite distributed delay and asymptotic stability was obtained with infinite distributed delay, but the paper did not consider the impulsive effects on the system.

Recently, to stabilize or synchronize nonlinear dynamical systems, impulsive control strategy has been widely concerned as an important control method. There are many interesting results about impulsive neural networks [4,8-9]. Therefore, we introduce the control model of system (1) represented by the following form:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \alpha_i(x_i(t))\Big[-\beta_i(x_i(t)) + \sum_{j=1}^{n} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}\mu_j + I_i \\ \qquad + \bigwedge_{j=1}^{n} a_{ij}\int_{t-\tau}^{t} K_{ij}(t-s) f_j(x_j(s))\,ds + \bigvee_{j=1}^{n} b_{ij}\int_{t-\tau}^{t} K_{ij}(t-s) f_j(x_j(s))\,ds \\ \qquad + \bigwedge_{j=1}^{n} T_{ij}\mu_j + \bigvee_{j=1}^{n} H_{ij}\mu_j\Big],\ t \neq t_k, t \geq t_0, \\ x_i(t_k^{+}) = p_{ik}(x(t_k)),\ t = t_k,\ i = 1,2,\cdots,n, k \in \mathbf{Z}^{+}. \end{cases} \tag{1}$$

for $i = 1,2,\cdots,n$; $k = 1,2,\cdots$, where $x_i(t)$ is the ith neuron state. $\alpha_i(x_i(t))$ is the amplification function, $\beta_i(x_i(t))$ represents an appropriately behaved function. For the details about f_j, a_{ij}, b_{ij}, c_{ij}, d_{ij}, T_{ij}, H_{ij}, μ_i and I_i, one can refer to [9,10]. $K_{ij}(s) \geq 0$ is the feedback kernel, defined on the interval $[0, +\infty]$ when τ is a positive finite number or $[0,\infty]$ while τ is infinite. Kernels are continuous and satisfy $\int_0^{\tau} K_{ij}(s)\,ds = 1, i,j = 1, 2,\cdots,n$. The time sequence $\{t_k\}$ is called impulsive moment and satisfies $0 < t_1 < t_2 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, and functions $p_{ik}(u) = p_{ik}(u_1,u_2,\cdots,u_n) \in \mathbf{C}[\mathbf{R}^n,\mathbf{R}]$ denote the external control inputs and satisfy the following condition:

(A) There exist nonnegative matrices $P_k = (p_{ij}^k)_{n\times n}$ such that

$$|p_{ik}(\bar{v}_1,\cdots,\bar{v}_n) - p_{ik}(v_1,\cdots,v_n)| \leq \sum_{j=1}^{n} p_{ij}^k |\bar{v}_j - v_j|,$$

for any $(\bar{v},\cdots,\bar{v}_n)^{T}, (v_1,\cdots,v_n)^{T} \in \mathbf{R}^n$, $i = 1,2,\cdots,n$, $k \in \mathbf{Z}^{+}$.

Remark 1 In this paper, we note that the impulsive functions may be nonlinear and even dependent on the states of all neurons. If impulsive functions $p_{ik}(x(t_k)) = p_{ik}(x_i(t_k)) + x_i(t_k)$ for all $i = 1,2,\cdots,n, k \in \mathbf{Z}^{+}$, condition (A) turns to the following form.

(A′) There exist nonnegative matrices $\bar{P}_k = (\bar{p}_{ij}^k)_{n\times n}$ such that

$$| p_k(u) - p_k(v) | \leq \bar{P}_k | u - v |,$$

for any $i = 1,2,\cdots,n, k \in \mathbf{Z}^{+}$, where $p_k(u) = (p_{1k}(u_1),\cdots,p_{nk}(u_n))^{T}$ and $\bar{p}_{ij}^k = (p_{ij}^k)^{\frac{1}{r}}$.

Such a function and assumptions have been required in [11]. Furthermore, if $p_{ik}(x(t_k)) = x_i(t_k) - r_{ik}x_i(t_k)$ for all $i = 1,2,\cdots,n, k \in \mathbf{Z}^{+}$, where $0 < r_{ik} < 2$, then it is just the condition which has been given in [12]. In view of this, our conditions of impulsive functions are more general.

The rest of this paper is organized as follows: In Section 2, we show some notations and definitions required, and state some preliminary results needed in later sections. In Section 3, we establish our main results by constructing a proper Lyapunov functional. At last, we give an example to verify the theoretical results.

1 Preliminaries

To obtain our results, we need the following assumptions.

(H$_1$) $0 < \underline{\alpha}_i \leqslant \alpha_i(x) \leqslant \bar{\alpha}_i$ ($\underline{\alpha}_i$ and $\bar{\alpha}_i$ are constants) for all $x \in \mathbf{R}$, $i = 1,2,\cdots,n$;

(H$_2$) For activation function $f_i(x)$, there exist $L_i > 0$ such that

$$L_i = \sup_{x \neq y} \left| \frac{f_i(x) - f_i(y)}{x - y} \right|$$

for all $x, y \in \mathbf{R}, x \neq y$, $i = 1,2,\cdots,n$;

(H$_3$) There exists a positive diagonal matrix $\beta = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_n)$ such that

$$\frac{\beta_i(x) - \beta_i(y)}{x - y} \geqslant \beta_i > 0, x \neq y, i = 1,2,\cdots,n;$$

(H$_4$) There exist constants $r \geqslant 1$ and $d_i > 0 (i = 1,2,\cdots,n)$, such that

$$\beta_i > \frac{r-1}{r} \sum_{j=1}^{n} (|c_{ij}| + |a_{ij}| + |b_{ij}|) L_j + \frac{1}{r} \sum_{j=1}^{n} \frac{d_j}{d_i} (|c_{ji}| + |a_{ji}| + |b_{ji}|) L_i.$$

In order to obtain our results, we introduce some notations and definitions.

For matrix $M = (m_{ij})_{n\times n} \in \mathbf{R}^{n\times n}$, $|M|$ denotes the absolute-value matrix given by $|M| = (|m_{ij}|)_{n\times n}$. For $v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}} \in \mathbf{R}^n$, $|v|$ denotes the absolute-value vector given by $|v| = (|v_1|, |v_2|, \cdots, |v_n|)^{\mathrm{T}}$. For any $v \in \mathbf{R}^n$, let vector norms $\|v\|_r$ and $\|v\|_\infty$ be defined as $\|v\|_r = (\sum_{i=1}^{n} |v_i|^r)^{\frac{1}{r}}$, $\|v\|_\infty = \max_{1\leqslant i \leqslant n} |v_i|$.

Definition 1 A function $x(t): [-\tau, +\infty) \to \mathbf{R}^n$ is called a solution of system (1) with the initial condition $x(s) = \varphi(s) \in \boldsymbol{PC}([-\tau, 0], \mathbf{R}^n)$, if $x(t)$ is continuous at $t \neq t_k$ and $t \geqslant 0$, $x(t_k) = x(t_k^+)$ and $x(t_k^-)$ exist, $x(t)$ satisfies model (1) for $t \geqslant 0$ under the initial condition.

Definition 2 For system (1), we call an equilibrium point $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^{\mathrm{T}}$ is globally exponentially stable, if for any solution $x(t)$ with the initial condition $\varphi \in \boldsymbol{PC}$, there exist constants $\varepsilon > 0$ and $M > 0$ such that

$$\|x_i(t) - x^*\|_r \leqslant M \|\varphi - x^*\|_{r\tau} e^{-\varepsilon t}$$

for all $t \geqslant 0$, where $\|\varphi - x^*\|_{r\tau} = \sup_{-\tau \leqslant \theta \leqslant 0} \|\varphi(\theta) - x^*\|_r$ and ε is called globally exponentially convergent rate.

To obtain our main results of this paper, we need the following lemmas.

Lemma 1 [2] For any $i \in \{1,2,\cdots,n\}$, suppose u and v are two states of system (1). Then we have

$$\left|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(u_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(v_j)\right| \leqslant \sum_{j=1}^{n} |\alpha_{ij}| |f_j(u_j) - f_j(v_j)|,$$

$$\left|\bigvee_{j=1}^{n} \beta_{ij} f_j(u_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(v_j)\right| \leqslant \sum_{j=1}^{n} |\beta_{ij}| |f_j(u_j) - f_j(v_j)|.$$

Lemma 2 [10] Let $a > 0, b > 0$ be constants, for any constant $p \geqslant 1$, we have

$$p a^{p-1} b \leqslant (p-1) a^p + b^p. \tag{2}$$

Lemma 3 [13] A locally invertible continuous mapping $H: \mathbf{R}^n \to \mathbf{R}^n$ is a homeomorphism of $\mathbf{R}^n$ onto itself if and only if $\|H(v_m)\| \to \infty (m \to \infty)$ as $\|v_m\| \to \infty (m \to \infty)$, for any $\{v_m\} \subset \mathbf{R}^n$.

2 Main Results

Now, we will show the main conclusions of this paper.

Theorem 1 Assume that (H$_1$) ~ (H$_4$) hold, then system (1) has a unique equilibrium point $x^* = (x_1^*,$

$x_2^*,\cdots,x_n^*)^{\mathrm T}$.

Proof Let $\tilde I_i = \sum_{j=1}^n d_{ij}\mu_j + \bigwedge_{j=1}^n T_{ij}\mu_j + \bigvee_{j=1}^n H_{ij}\mu_j + I_i, i = 1,2,\cdots,n$. We assume that $x^* = (x_1^*,x_2^*,\cdots,x_n^*)^{\mathrm T}$ is an equilibrium point of system(1), then, for $i = 1,2,\cdots,n$, x^* satisfies the following equation:

$$\beta_i(x_i^*) = \sum_{j=1}^n c_{ij} f_j(x_j^*) + \bigwedge_{j=1}^n a_{ij} f_j(x_j^*) + \bigvee_{j=1}^n b_{ij} f_j(x_j^*) + \tilde I_i .$$

Let $H(x) = (H_1(x),H_2(x),\cdots,H_n(x))^{\mathrm T}$, where, for $i = 1,2,\cdots,n$,

$$H_i(x) = -\beta_i(x_i) + \sum_{j=1}^n c_{ij} f_j(x_j) + \bigwedge_{j=1}^n a_{ij} f_j(x_j) + \bigvee_{j=1}^n b_{ij} f_j(x_j) + \tilde I_i .$$

In the following, we shall prove that $H(x)$ is a homeomorphism of $\mathbf{R}^n$ onto itself.

1) We show that $H(x)$ is an injective map on $\mathbf{R}^n$.

In fact, if there exist $x = (x_1,x_2,\cdots,x_n)^{\mathrm T}, y = (y_1,y_2,\cdots,y_n)^{\mathrm T} \in \mathbf{R}^n$, and $x \neq y$ such that $H(x) = H(y)$, then, for $i = 1,2,\cdots,n$,

$$\beta_i(y_i) - \beta_i(x_i) = \sum_{j=1}^n c_{ij}[f_j(y_j) - f_j(x_j)] + [\bigwedge_{j=1}^n a_{ij} f_j(y_j) - \bigwedge_{j=1}^n a_{ij} f_j(x_j)] + [\bigvee_{j=1}^n b_{ij} f_j(y_j) - \bigvee_{j=1}^n b_{ij} f_j(x_j)] .$$

By (H_2), (H_3) and Lemma 1, we have, for $i = 1,2,\cdots,n$,

$$\beta_i |y_i - x_i| \leqslant \sum_{j=1}^n (|c_{ij}| + |a_{ij}| + |b_{ij}|) L_j |y_j - x_j| .$$

In view of Lemma 2 and the above inequality, take $a = \dfrac{|y_i - x_i|}{|y_j - x_j|^{\frac{1}{r}}}, b = |y_j - x_j|^{\frac{r-1}{r}}$, then there exist $d_i > 0, r \geqslant 1$ such that, for $i = 1,2,\cdots,n$,

$$d_i\beta_i r|y_i - x_i|^r = d_i\beta_i|y_i - x_i| r\left(\frac{|y_i - x_i|}{|y_j - x_j|^{\frac{1}{r}}}\right)^{r-1} |y_j - x_j|^{\frac{r-1}{r}}$$
$$\leqslant d_i \sum_{j=1}^n L_j(|c_{ij}| + |a_{ij}| + |b_{ij}|)[(r-1)|y_i - x_i|^r + |y_j - x_j|^r] .$$

Take sum of both sides of the above inequality, we have

$$\sum_{i=1}^n r d_i\beta_i |x_i - y_i|^r \leqslant \sum_{i=1}^n d_i\{\sum_{j=1}^n L_j(|c_{ij}| + |a_{ij}| + |b_{ij}|)[(r-1)|y_i - x_i|^r + |y_j - x_j|^r]$$
$$= \sum_{i=1}^n d_i\{(r-1)\sum_{j=1}^n [(|c_{ij}| + |a_{ij}| + |b_{ij}|) L_j] + \sum_{i=1}^n [(|c_{ji}| + |a_{ji}| + |b_{ji}|) L_i]\}\cdot |y_i - x_i|^r.$$

Hence,

$$\sum_{i=1}^n d_i r\{\beta_i - \frac{r-1}{r}\sum_{j=1}^n (|c_{ij}| + |a_{ij}| + |b_{ij}|) L_j - \frac{1}{r}\sum_{j=1}^n \frac{d_j}{d_i}(|c_{ji}| + |a_{ji}| + |b_{ji}|) L_i\} |y_i - x_i|^r \leqslant 0. \quad (3)$$

By(H_4), we can see that(3) holds if and only if $x_i = y_i (i = 1,2,\cdots,n)$. Therefore, $x = y$.

2) We prove that $\|H(x_m)\| \to +\infty$ as $\|x_m\| \to +\infty$, for any $\{x_m\}$ of $\mathbf{R}^n$, or equivalently, we need to check that $\|H(x_m) - H(0)\| \to \infty$ as $m \to \infty$.

If it is not true, there exists a subsequence of $\{x_m\}$ (without loss of generality, we take $\{x_m\}$ itself), such that $\{\|H(x_m)\| - \|H(0)\|\}$ is bounded, i.e., for some constant $N_0 > 0$,

$$|H_i(x_m) - H_i(0)| \leqslant N_0, i = 1,2,\cdots,n, m = 1,2,\cdots, \quad (4)$$

where $H_i(x_m)$ is the ith component of $H(x_m)$. Denote x_m^i as the ith component of x_m, we get, for $i = 1,2,\cdots,n$,

$$\beta_i x_m^i + H_i(x_m) - H_i(0) \leqslant \sum_{j=1}^n c_{ij}[f_j(x_m^j) - f_j(0)] + \bigwedge_{j=1}^n a_{ij}[f_j(x_m^j) - f_j(0)] + \bigvee_{j=1}^n b_{ij}[f_j(x_m^j) - f_j(0)] . \quad (5)$$

Furthermore,

$$\beta_i|x_m^i| - |H_i(x_m) - H_i(0)| \leqslant \sum_{j=1}^{n} L_j(|c_{ij}| + |a_{ij}| + |b_{ij}|)|x_m^j|, i = 1,2,\cdots,n.$$

Multiply both sides of the above inequality by $rd_i|x_m^i|^{r-1}$, from Lemma 2, then, for $i = 1,2,\cdots,n$,

$$\beta_i r_i|x_m^i|^r - |H_i(x_m) - H_i(0)|rd_i|x_m^i|_i^{r-1} \leqslant d_i\sum_{j=1}^{n}[(|c_{ij}| + |a_{ij}| + |b_{ij}|)L_j][(r-1)|x_m^i|^r + |x_m^j|^r].$$

Take sum of both sides of the above inequality to get

$$\sum_{i=1}^{n} d_i r\Big\{\beta_i - \frac{r-1}{r}\sum_{j=1}^{n}[(|c_{ij}| + |a_{ij}| + |b_{ij}|)L_j - \frac{1}{r}\sum_{j=1}^{n}\frac{d_j}{d_i}[(|c_{ji}| + |a_{ji}| + |b_{ji}|)L_i\Big\}|x_m^i|^r$$

$$\leqslant \sum_{j=1}^{n} d_i r|h_i(x_m) - h_i(0)||x_m^i|^{r-1}.$$

By(H_4) and(4), the above inequality yields

$$r_0\sum_{i=1}^{n}|x_m^i|^r \leqslant \sum_{i=1}^{n} d_i r N_0|x_m^i|^{r-1} \leqslant N|x_m^i|^{r-1}, \tag{6}$$

where: $N = \max_{1\leqslant i\leqslant n}\{d_i r N_0\}$ and

$$r_0 = \min_{1\leqslant i\leqslant n}\{d_i r[\beta_i - \frac{r-1}{r}\sum_{j=1}^{n}[(|c_{ij}| + |a_{ij}| + |b_{ij}|)L_j - \frac{1}{r}\sum_{j=1}^{n}\frac{d_j}{d_i}[(|c_{ji}| + |a_{ji}| + |b_{ji}|)L_i\} > 0.$$

Since all the kinds of norms defined in $\mathbf{R}^n$ are equivalent, there is a constant $C > 0$, such that $\|x\|_\infty \leqslant C\|x\|_p$, $x \in \mathbf{R}^n$.

By(6), we obtain $r_0\|x_m\|_r^r \leqslant nN\|x_m\|_\infty^{r-1} \leqslant nNC^{r-1}\|x_m\|_p^{r-1}$, i. e.,

$$\|x_m\|_r \leqslant \frac{nN}{r_0}C^{r-1} < \infty,$$

which is a contradiction to $\|x_m\| \to \infty$ as $m \to \infty$. So $\|H(x_m)\| \to \infty\ (m \to \infty)$. By Lemma 3, we see that $H(x)$ is a homeomorphism. Hence, for system(1), we conclude that there is a unique equilibrium point x^*.

Now, we consider the function

$$\xi_i(x) = \frac{x}{\bar{\alpha}_i} - \beta_i + \frac{r-1}{r}[\sum_{j=1}^{n}|c_{ij}|L_j + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)L_j\int_0^\tau K_{ij}(s)e^{xs}ds]$$

$$+ \frac{1}{r}\sum_{j=1}^{n}\frac{d_j}{d_i}[|c_{ji}|L_i + \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)L_i\int_0^\tau K_{ji}(s)e^{xs}ds],$$

where: $i = 1,2,\cdots,n$.

From(H_4), we know that $\xi_i(0) < 0$. Obviously, $\xi_i(x)$ is continuous. Also, we have $\xi_i(x) \to +\infty$ as $x \to +\infty$. So there exist constants δ_i such that $\xi_i(\delta_i) = 0$ for $i = 1,2,\cdots,n$.

Choose $\delta < \min\{\delta_1,\delta_2,\cdots,\delta_n\}$, then for $i = 1,2,\cdots,n$, we have

$$\frac{\delta}{\bar{\alpha}_i} - \beta_i + \frac{r-1}{r}[\sum_{j=1}^{n}|c_{ij}|L_j + \sum_{j=1}^{n}(|a_{ij}| + |b_{ij}|)L_j\int_0^\tau K_{ij}(s)e^{\delta s}ds]$$

$$+ \frac{1}{r}\sum_{j=1}^{n}\frac{d_j}{d_i}[|c_{ji}|L_i + \sum_{j=1}^{n}(|a_{ji}| + |b_{ji}|)L_i\int_0^\tau K_{ji}(s)e^{\delta s}ds] \leqslant 0. \tag{7}$$

In the following, we will prove that this unique equilibrium point x^* of system(1) is globally exponentially stable.

Theorem 2 Assume that (H_1) ~ (H_4),(A) hold. Furthermore, suppose the following condition is satisfied:

(C) There exists a constant $\alpha \geqslant 0$ such that $\frac{\ln\rho_k}{t_k - t_{k-1}} \leqslant \alpha < \delta, k \in \mathbf{Z}^+$, where: $\rho_k = \max_{1\leqslant i\leqslant n}\{1, \sum_{j=1}^{n} p_{ij}^k\}$, δ is defined in(7).

Then the unique equilibrium point x^* of system(1) with finite distributed delay is globally exponentially stable with the exponential convergence rate $\delta - \frac{\alpha}{r}$.

Proof Let $y_i(t) = x_i(t) - x_i^*$, then from(5), for $i = 1,2,\cdots,n, k \in \mathbf{Z}^+$, we have

$$\begin{cases} \dfrac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \alpha_i(x_i(t))\Big[-(\beta_i(x_i(t)) - \beta_i(x_i^*)) + \sum_{j=1}^{n} c_{ij}(f_j(x_j(t)) - f_j(x_j^*)) \\ \quad + (\bigwedge_{j=1}^{n} a_{ij}\int_{t-\tau}^{t} K_{ij}(t-s)f_j(x_j(s))\,\mathrm{d}s - \bigwedge_{j=1}^{n} a_{ij}\int_{t-\tau}^{t} K_{ij}(t-s)f_j(x_j^*)\,\mathrm{d}s) \\ \quad + (\bigvee_{j=1}^{n} b_{ij}\int_{t-\tau}^{t} K_{ij}(t-s)f_j(x_j(s))\,\mathrm{d}s - \bigvee_{j=1}^{n} b_{ij}\int_{t-\tau}^{t} K_{ij}(t-s)f_j(x_j^*)\,\mathrm{d}s)\Big], t \neq t_k, \\ y_i(t_k^+) = p_{ik}(x(t_k)) - p_{ik}(x^*), t = t_k, i = 1,2,\cdots,n, k \in \mathbf{Z}^+. \end{cases} \tag{8}$$

By considering the first equation of system(8), we have

$$D^+|y_i(t)| \leqslant \alpha_i(x_i(t))\Big[-\beta_i|y_i(t)| + \sum_{j=1}^{n} L_j|c_{ij}||y_j(t)| + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)|y_j(t-s)|\,\mathrm{d}s\Big],$$

for all $t > t_0, t \neq t_k, i = 1,2,\cdots,n, k = 1,2,\cdots$.

Let $W_i(t) = \mathrm{e}^{\delta t}|y_i(t)|, i = 1,2,\cdots,n$, then

$$D^+W_i(t) \leqslant \delta W_i(t) + \alpha_i(x_i(t))\Big[-\beta_i W_i(t) + \sum_{j=1}^{n} L_j|c_{ij}|W_j(t) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}W_j(t-s)\,\mathrm{d}s\Big]$$

$$\leqslant \alpha_i(x_i(t))\Big[(\frac{\delta}{\alpha_i} - \beta_i)W_i(t) + \sum_{j=1}^{n} L_j|c_{ij}|W_j(t) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}W_j(t-s)\,\mathrm{d}s\Big],$$

for $t > t_0, t \neq t_k, i = 1,2,\cdots,n, k = 1,2,\cdots$.

Multiply both sides of the above inequality by $rW_i^{r-1}(t)$, by Lemma 2, we have

$$\begin{aligned} \int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}W_j(t-s)rW_i^{r-1}(t)\,\mathrm{d}s\Big] &\leqslant \alpha_i(x_i(t))\Big[(\frac{\delta}{\alpha_i} - \beta_i)W_i^r(t) + \sum_{j=1}^{n} L_j|c_{ij}|[(r-1)W_i^r(t) + W_j^r(t)] \\ &\quad + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}[(r-1)W_i^r(t) + W_j^r(t-s)]\,\mathrm{d}s\Big] \\ &= \alpha_i(x_i(t))\{r(\frac{\delta}{\alpha_i} - \beta_i)W_i^r(t) + (r-1)[\sum_{j=1}^{n} L_j|c_{ij}| + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\cdot \\ &\quad \int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}\mathrm{d}s]W_i^r(t) + \sum_{j=1}^{n} L_j|c_{ij}|W_j^r(t) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\cdot \\ &\quad \int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}W_j^r(t-s)\,\mathrm{d}s\}. \end{aligned}$$

Now we construct the Lyapunov functional

$$V(t) = \sum_{i=1}^{n} d_i\Big[\frac{1}{\alpha_i}W_i^r(t) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}(\int_{t-s}^{t} W_j^r(\xi)\,\mathrm{d}\xi)\,\mathrm{d}s\Big].$$

Then, we obtain

$$D^+V(t) = \sum_{i=1}^{n} d_i\Big[\frac{1}{\alpha_i}rW_i^{r-1}(t)D^+W_i(t) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|)\int_0^{\tau} K_{ij}(s)\mathrm{e}^{\delta s}(W_j^r(t) - W_j^r(t-s))\,\mathrm{d}s\Big]$$

$$\leqslant \sum_{i=1}^{n} d_i \{ r \frac{\alpha_i(x_i(t))}{\bar{\alpha}_i} [(\frac{\delta}{\bar{\alpha}_i} - \beta_i) + \frac{r-1}{r} (\sum_{j=1}^{n} L_j |c_{ij}| + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|) \int_0^{\tau} K_{ij}(s) e^{\delta s} ds)$$

$$+ \frac{1}{r} \sum_{j=1}^{n} \frac{d_j}{d_i} (L_i |c_{ji}| + \sum_{j=1}^{n} L_i(|a_{ji}| + |b_{ji}|) \int_0^{\tau} K_{ji}(s) e^{\delta s} ds)] \} W_i^r(t) + \sum_{i=1}^{n} d_i (\frac{\alpha_i(x_i(t))}{\bar{\alpha}_i}$$

$$- 1) \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|) \int_0^{\tau} K_{ij}(s) e^{\delta s} W_j^r(t-s) ds$$

$$\leqslant \sum_{i=1}^{n} d_i \{ r \frac{\alpha_i(x_i(t))}{\bar{\alpha}_i} [(\frac{\delta}{\bar{\alpha}_i} - \beta_i) + \frac{r-1}{r} (\sum_{j=1}^{n} L_j |c_{ij}| + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|) \int_0^{\tau} K_{ij}(s) e^{\delta s} ds)$$

$$+ \frac{1}{r} \sum_{j=1}^{n} \frac{d_j}{d_i} (L_i |c_{ji}| + \sum_{j=1}^{n} L_i(|a_{ji}| + |b_{ji}|) \int_0^{\tau} K_{ji}(s) e^{\delta s} ds)] \} W_i^r(t)$$

$$\leqslant 0.$$

Then, for $t \in (t_{k-1}, t_k]$, we have

$$V(t) \leqslant V(t_{k-1}^+), k \in \mathbf{Z}^+. \tag{9}$$

Meantime, from conditions (A) and Theorem 2 (C), we have

$$W_i^r(t_k^+) = e^{r\delta t_k^+} |p_{ik}(x(t_k)) - p_{ik}(x^*)|^r \leqslant e^{r\delta t_k} \sum_{j=1}^{n} p_{ij}^k |x_i(t_k) - x_i^*|^r \leqslant \rho_k e^{r\delta t_k} |x_i(t_k) - x_i^*|^r = \rho_k W_i^r(t_k).$$

Then

$$V(t_k^+) = \sum_{i=1}^{n} d_i [\frac{1}{\bar{\alpha}_i} W_i^r(t_k^+) + \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|) \int_0^{\tau} K_{ij}(s) e^{\delta s} (\int_{t_k^+ - s}^{t_k^+} W_j^r(\xi) d\xi) ds]$$

$$\leqslant \sum_{i=1}^{n} d_i [\frac{1}{\bar{\alpha}_i} \rho_k W_i^r(t_k) + \rho_k \sum_{j=1}^{n} L_j(|a_{ij}| + |b_{ij}|) \int_0^{\tau} K_{ij}(s) e^{\delta s} (\int_{t_k - s}^{t_k} W_j^r(\xi) d\xi) ds]$$

$$= \rho_k V(t_k). \tag{10}$$

From (9) and (10), we obtain

$$V(t) \leqslant V(t_{k-1}^+) \leqslant \rho_{k-1} V(t_{k-1}) \leqslant \rho_0 \rho_1 \cdots \rho_{k-1} V(0), \tag{11}$$

for $t \in (t_{k-1}, t_k], k \in \mathbf{Z}^+$, where $\rho_0 = 1, t_0 = 0$. According to condition Theorem 2 (C), we have

$$\rho_k \leqslant e^{\alpha(t_k - t_{k-1})}, k \in \mathbf{Z}^+.$$

It follows from (11) that

$$V(t) \leqslant e^{\alpha(t_1 - 0)} \cdots e^{\alpha(t_{k-1} - t_{k-2})} V(0) \leqslant e^{\alpha t} V(0), \text{ where: } t \in (t_{k-1}, t_k], k \in \mathbf{Z}^+.$$

So, we have

$$\sum_{i=1}^{n} \frac{d_i}{\bar{\alpha}_i} W_i^r(t) \leqslant V(t) \leqslant V(0) e^{\alpha t},$$

i. e.,

$$\sum_{i=1}^{n} \frac{d_i}{\bar{\alpha}_i} e^{r\delta t} | x_i(t) - x_i^* |^r \leqslant V(0) e^{\alpha t} \leqslant \zeta \|\varphi - x^*\|_{r\tau}^r e^{\alpha t},$$

where: ζ is a positive constant.

Therefore, $\|x(t) - x^*\|_r \leqslant M \|\varphi - x^*\|_{r\tau} e^{-(\delta - \frac{\alpha}{r})t}$, where: $M = [\frac{\zeta}{\min_{1 \leqslant i \leqslant n} \{d_i / \bar{\alpha}_i\}}]^{\frac{1}{r}}, t \geqslant 0$.

Obviously, we can easily obtain the following results.

In Theorem 2, we take $r = 1$ and $r = 2$, respectively, then we have the following results.

Corollary 1 Assume that $(H_1) \sim (H_3)$, (A) and Theorem 2 (C) hold, moreover, there exist constants

$d_i > 0$ such that

$$d_j \beta_j > \sum_{i=1}^{n} d_i L_j (|c_{ij}| + |a_{ij}| + |b_{ij}|), j = 1, 2, \cdots, n,$$

then the equilibrium point x^* of system (1) with finite delay is globally exponentially stable.

Corollary 2 Assume that $(H_1) \sim (H_3)$, (A) and Theorem 2 (C) hold, moreover, there exist constants $d_i > 0$ such that

$$2d_i \beta_i > \sum_{j=1}^{n} d_i (|c_{ij}| + |a_{ij}| + |b_{ij}|) L_j + \sum_{j=1}^{n} d_j (|c_{ji}| + |a_{ji}| + |b_{ji}|) L_i, i = 1, 2, \cdots, n,$$

then the equilibrium point x^* of system (1) with finite delay is globally exponentially stable.

Remark 3 From the conditions of Corollary 2, one can see that $\beta - L(|C| + |A| + |B|)$ is a nonsingular M – matrix, where $|A|, |B|, |C|$ are defined as above. So, our results contain the main results in [3] and other similar results.

Now, we give the result on the stability of FCGNNs with unbounded distributed delay.

Theorem 3 Assume that $(H_1) \sim (H_4)$, (A) and Theorem 2 (C) hold, then the unique equilibrium point x^* of system (1) with infinite delay is globally exponentially stable under the condition that

(H_5) for the feedback kernels, there exists positive number δ such that

$$\int_0^{+\infty} e^{\delta s} K_{ij}(s) \, ds < \infty, \text{ for any } i, j = 1, 2, \cdots, n.$$

Proof We omit it since it is similar to the proof of Theorem 1 and Theorem 2.

3 An Example

In this section, we give an illustrated example to verify our theoretical results given in Section 2.

Example 1 Let $n = 2$, we consider the following FCGNN with infinite distributed delays:

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \alpha_i(x_i(t)) \Big[-\beta_i(x_i(t)) + \sum_{j=1}^{2} c_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} d_{ij} \mu_j + I_i \\ \qquad + \bigwedge_{j=1}^{2} a_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s)) \, ds + \bigvee_{j=1}^{2} b_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s)) \, ds \\ \qquad + \bigwedge_{j=1}^{2} T_{ij} \mu_j + \bigvee_{j=1}^{2} H_{ij} \mu_j \Big], t \neq t_k, t \geq t_0, \\ x_i(t_k^+) = p_{ik}(x(t_k)), t = t_k, i = 1, 2, k \in \mathbf{Z}^+, \end{cases}$$

where: $\alpha_1(x_1(t)) = 2 + \sin(2x_1(t))$, $\alpha_2(x_2(t)) = 2 - \cos(x_2(t))$, $\beta_1(x_1(t)) = 3x_1(t)$, $\beta_2(x_2(t)) = 4x_2(t)$, $a_{11} = 0.07$, $a_{12} = 0.03$, $a_{21} = 0.06$, $a_{22} = 0.06$, $b_{11} = 0.01$, $b_{12} = 0.05$, $b_{21} = 0.03$, $b_{22} = 0.01$, $I_1 = I_2 = 0.01$, $c_{11} = 0.05$, $c_{12} = c_{21} = 0.02$, $c_{22} = 0.03$, $d_{11} = 0.03$, $d_{12} = 0.04$, $d_{21} = 0.02$, $d_{22} = 0.04$, $\mu_1 = \mu_2 = 0.01$, $T_{11} = 0.15$, $T_{12} = 0.17$, $T_{21} = 0.14$, $T_{22} = 0.15$, $H_{11} = 0.03$, $H_{12} = 0.07$, $H_{21} = 0.03$, $H_{22} = 0.14$, $\gamma_k = \eta_k = 1$, $f_i(x) = 0.5(|x+1| - |x-1|)$, $i = 1, 2$, $p_{1k}(x(t_k)) = -\gamma_k x(t_k^-)$, $p_{2k}(x(t_k)) = -\eta_k x(t_k^-)$,

$$K_{ij}(s) = \begin{cases} e^{-s}, 0 \leq s \leq 10 \\ 0, s > 10 \end{cases}, i, j = 1, 2.$$

By calculation, all the assumptions in Theorems 3 are satisfied. The initial condition is $x(t) = [-1.1, 1.1]^T$. The time series of the state component are showed in Figure 1. Therefore, the equilibrium point of system (1) with infinite delay is globally exponentially stable.

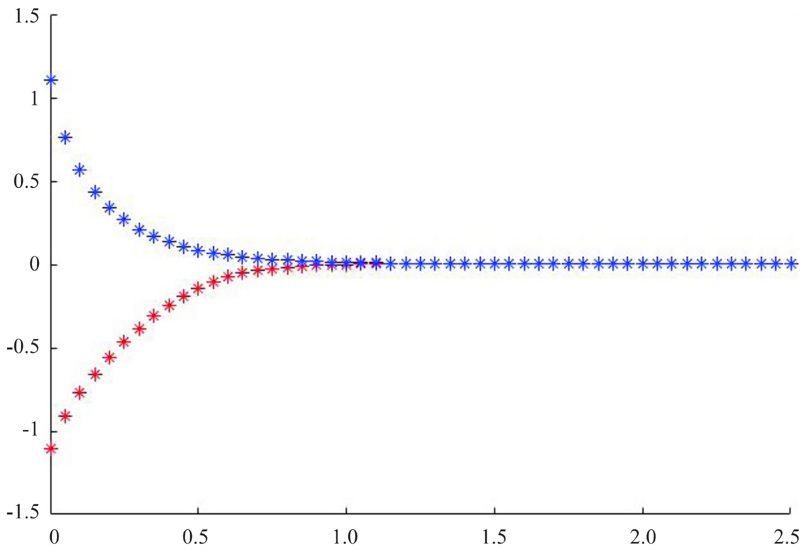

Fig.1 Behavior of the state component for $x_1(t)$, $x_2(t)$

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